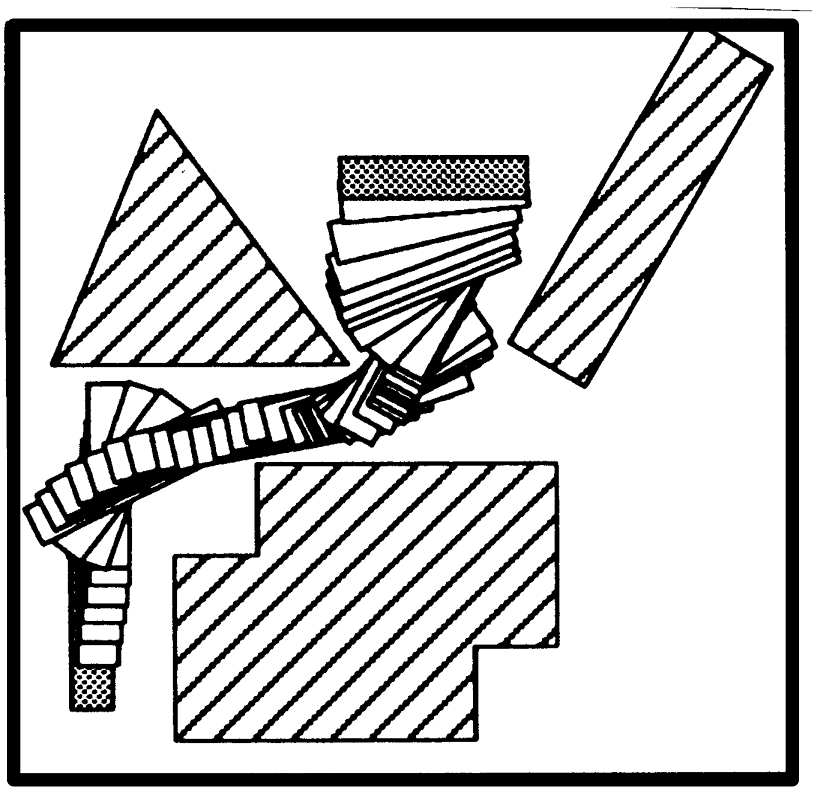
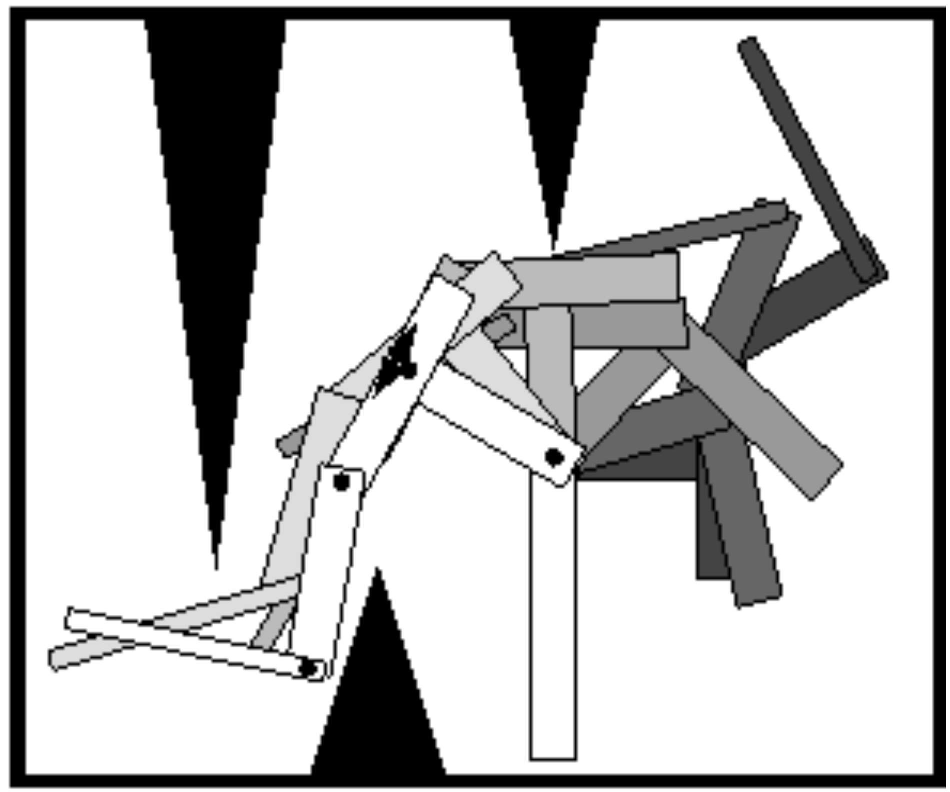
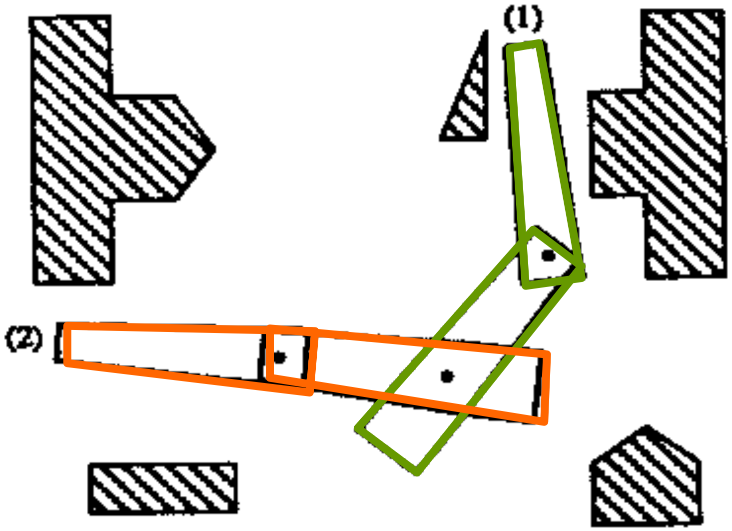
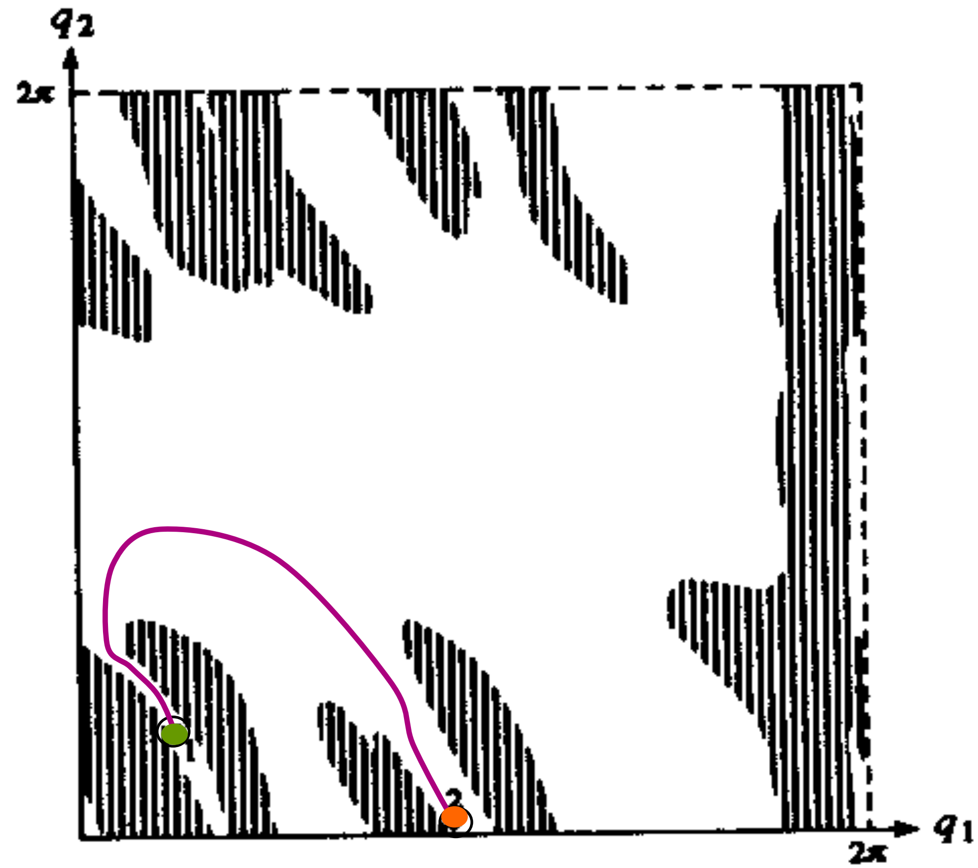
What is a Path?



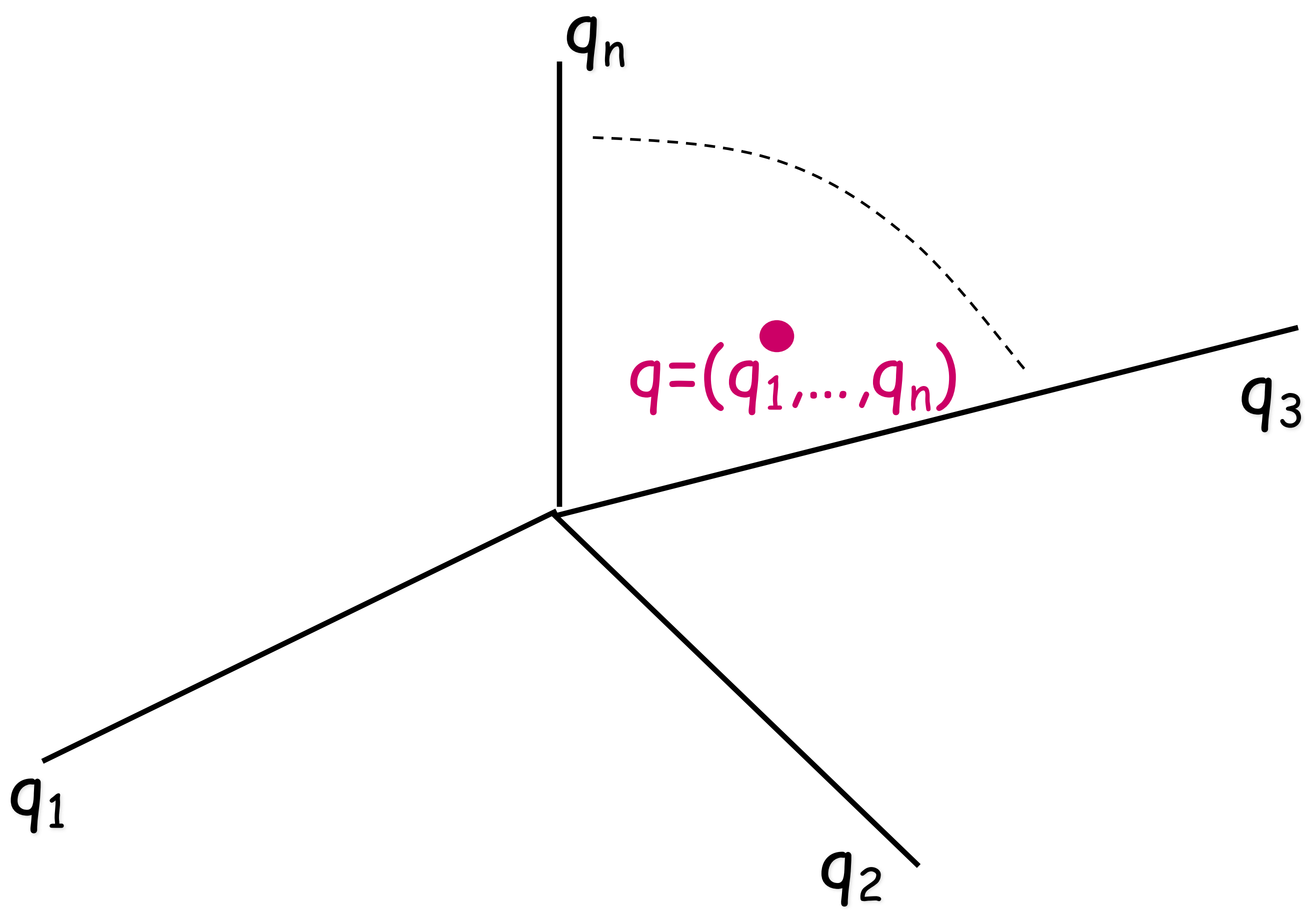


Tool: Configuration Space
(C-Space C)





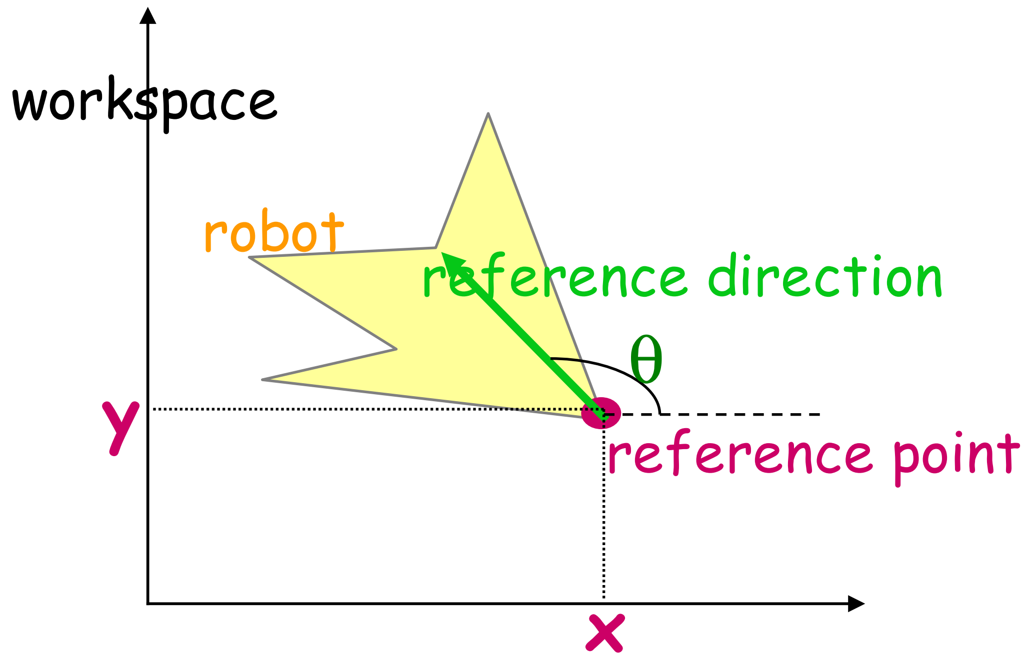
Configuration Space



Definition

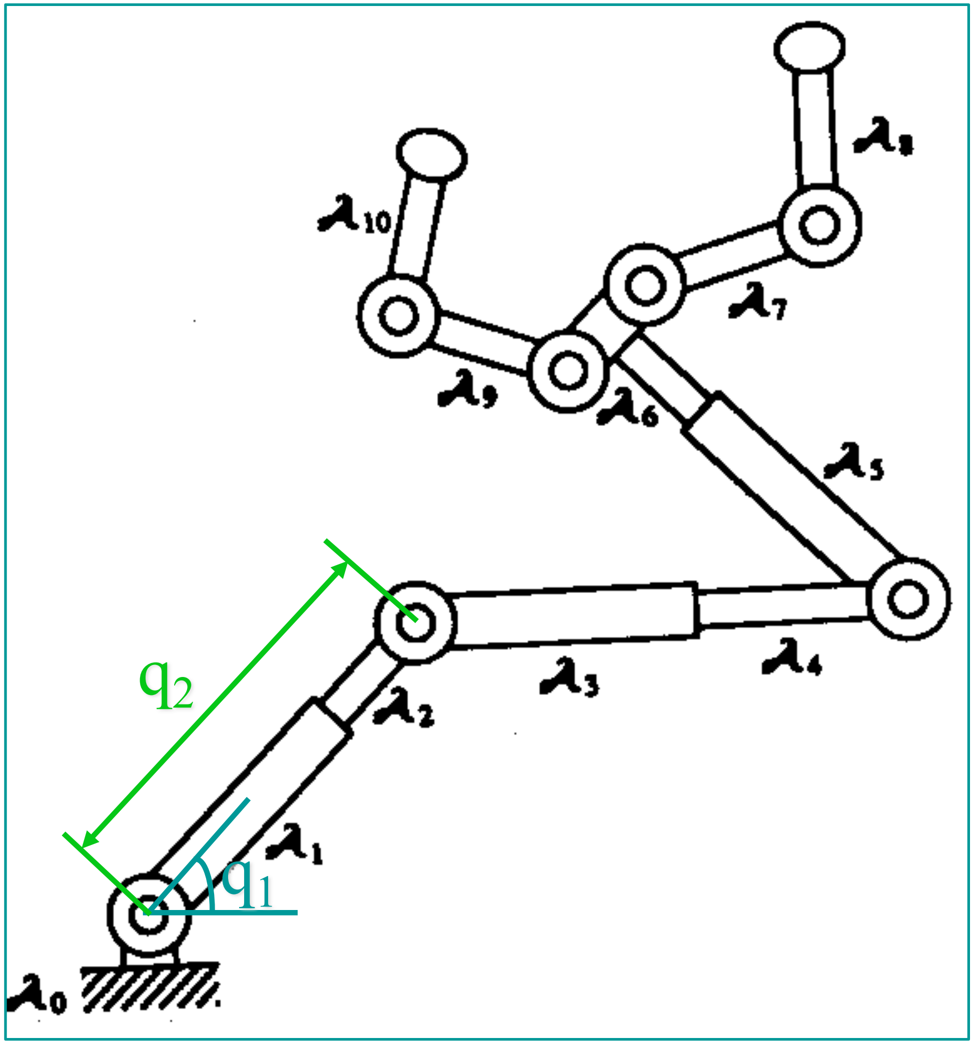
- A robot configuration is a specification of the positions of all robot points relative to a fixed coordinate system
- Usually a configuration is expressed as a "vector" of position/orientation parameters

Rigid Robot Example



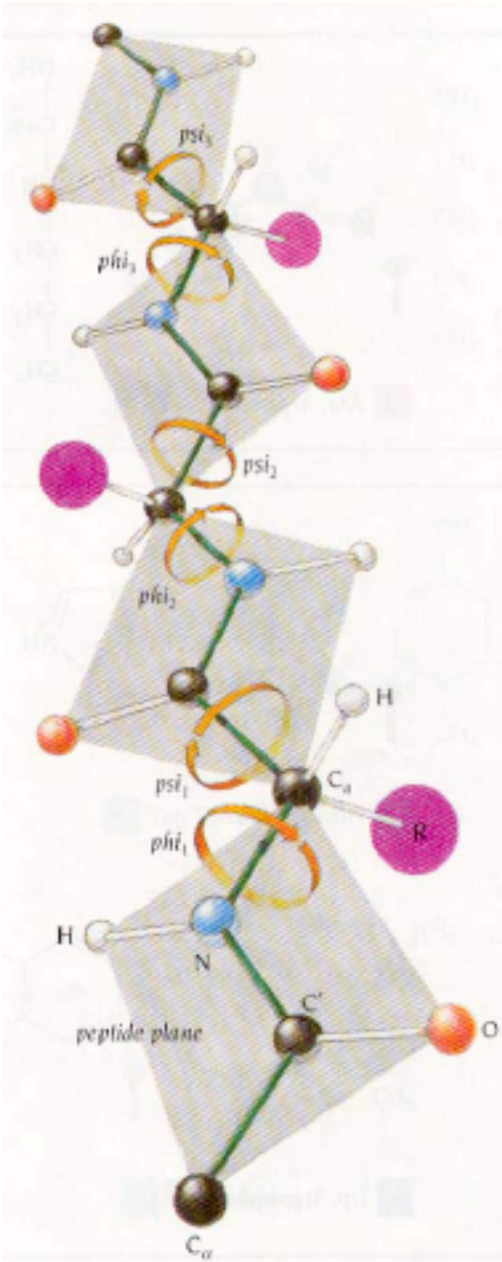
- 3-parameter representation: $q = (x, y, \theta)$
- In a 3-D workspace q would be of the form $(x, y, z, \alpha, \beta, \gamma)$

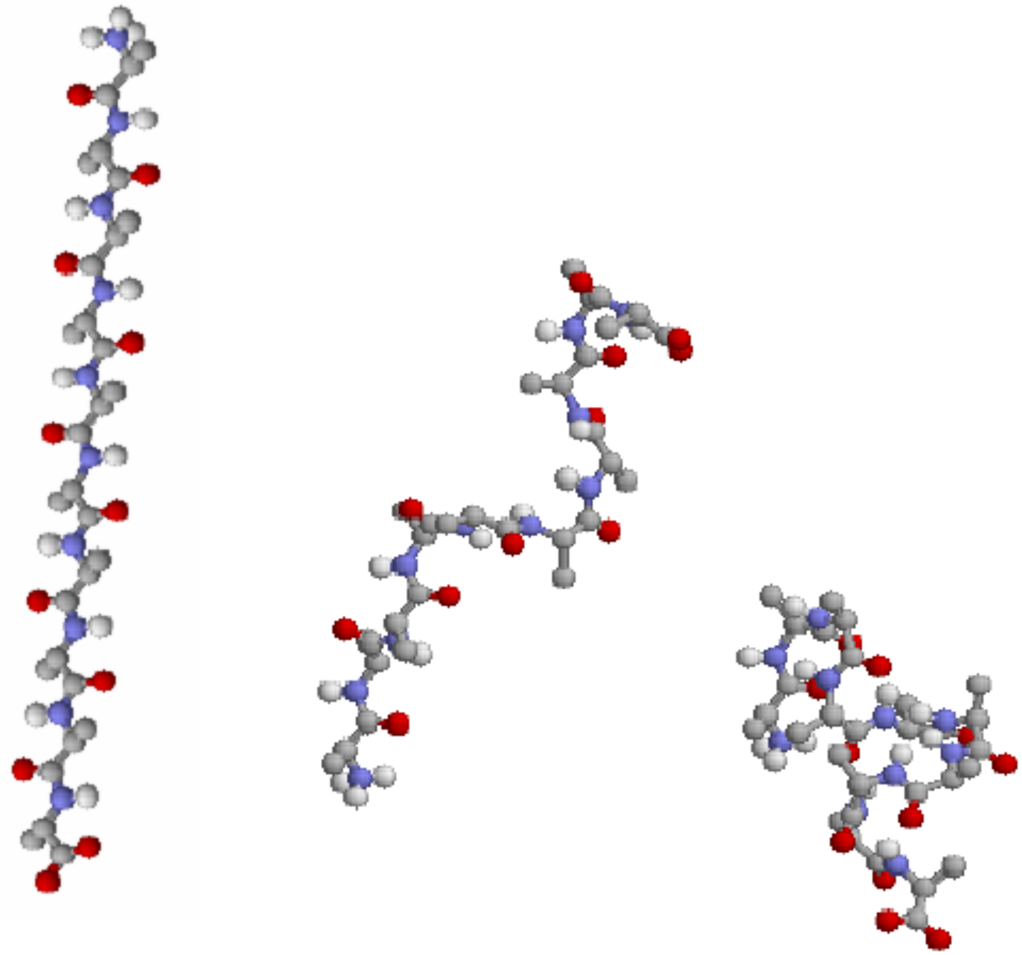
Articulated Robot Example



$$q = (q_1, q_2, \ldots, q_{10})$$

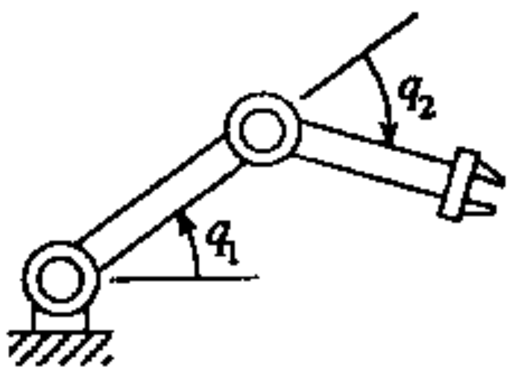
Protein example





Configuration Space of a Robot

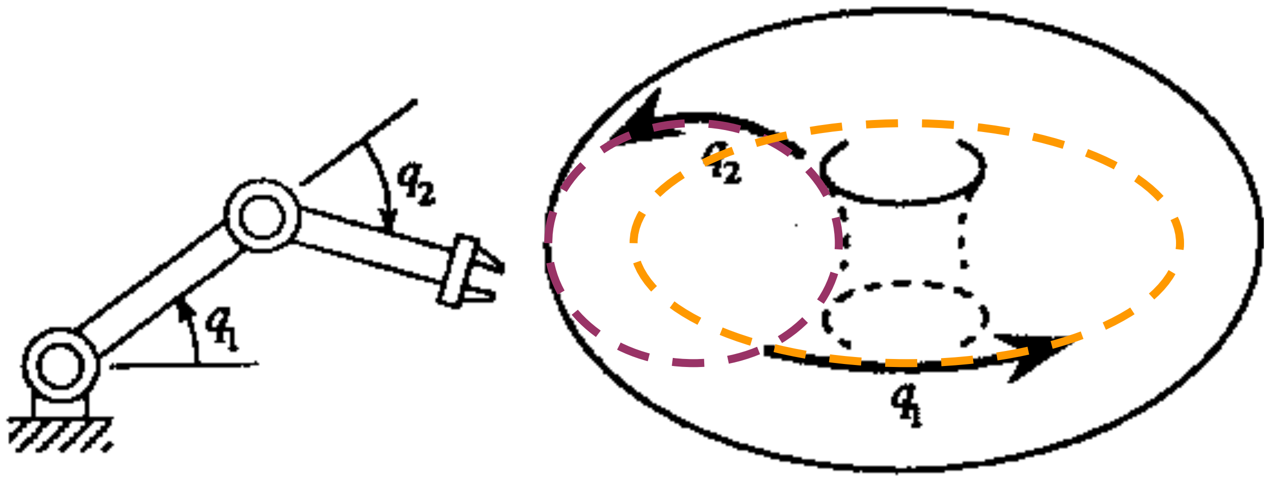
- Space of all its possible configurations
- But the topology of this space is usually not that of a Cartesian space



Configuration Space of a Robot

- Space of all its possible configurations
- But the topology of this space is usually not that of a Cartesian space

$C = S^1 \times S^1$



Configuration Space of a Robot

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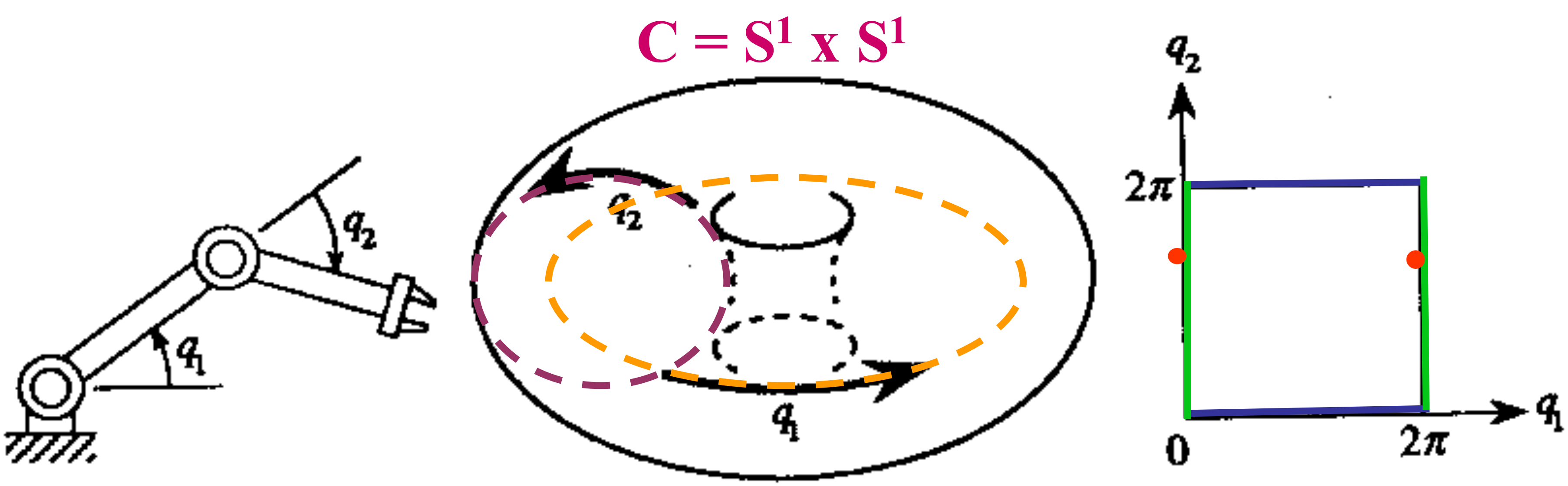
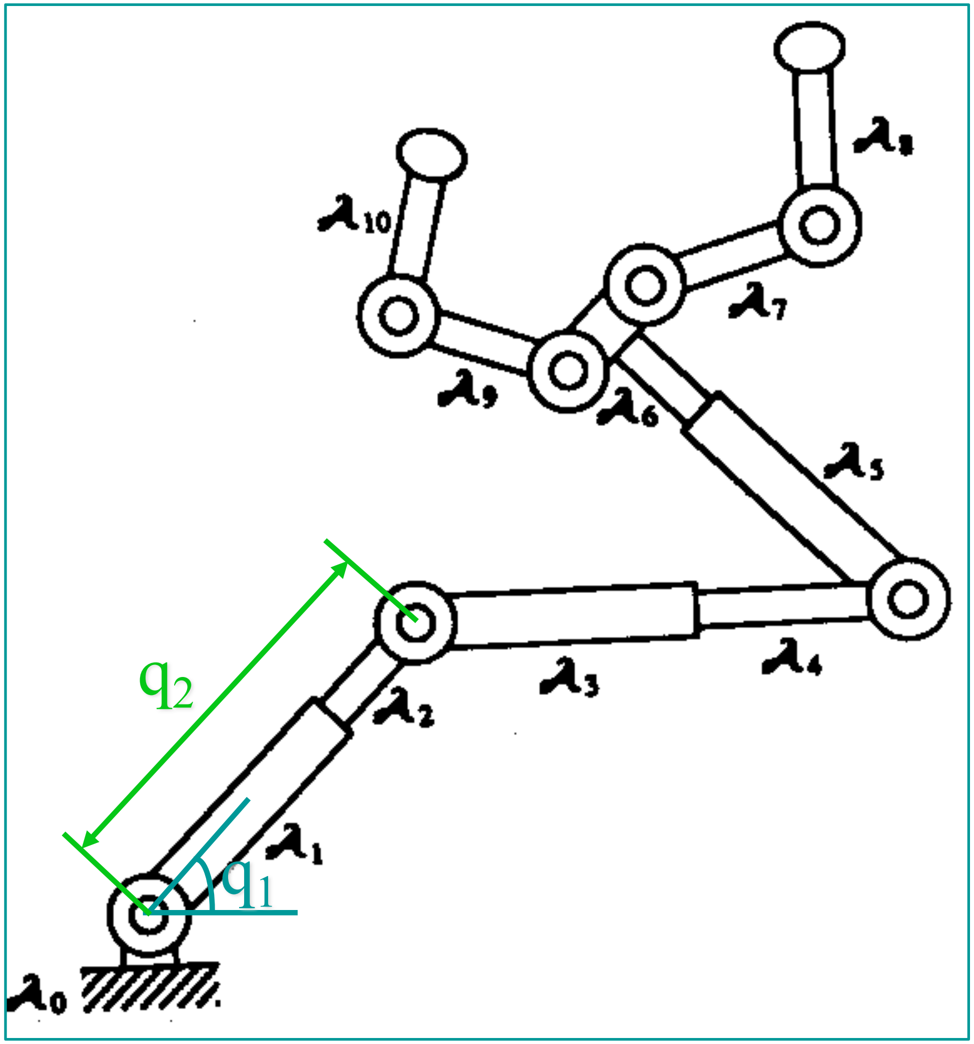


What is its Topology?

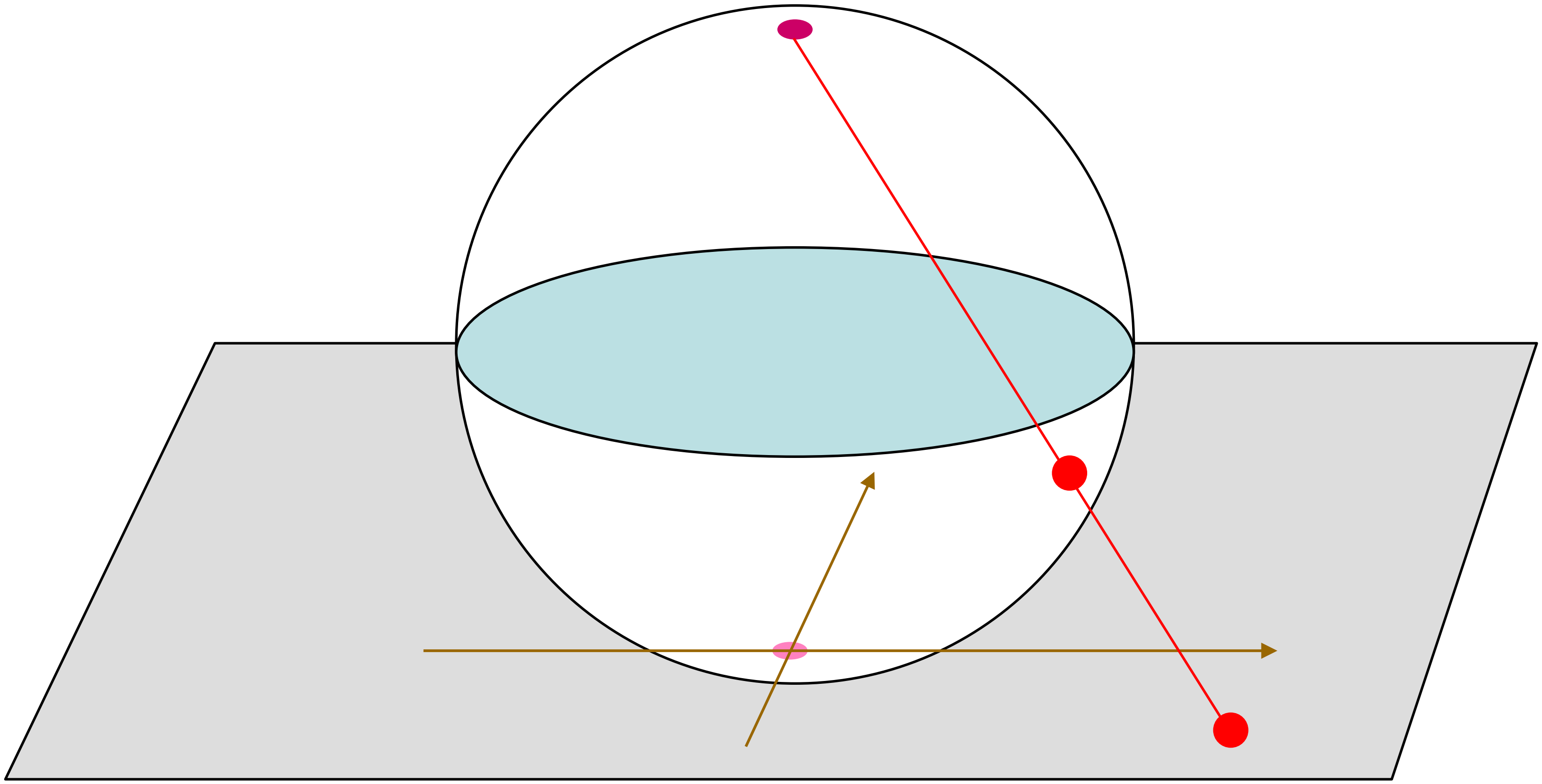


$(S1)^7 \times R^3$

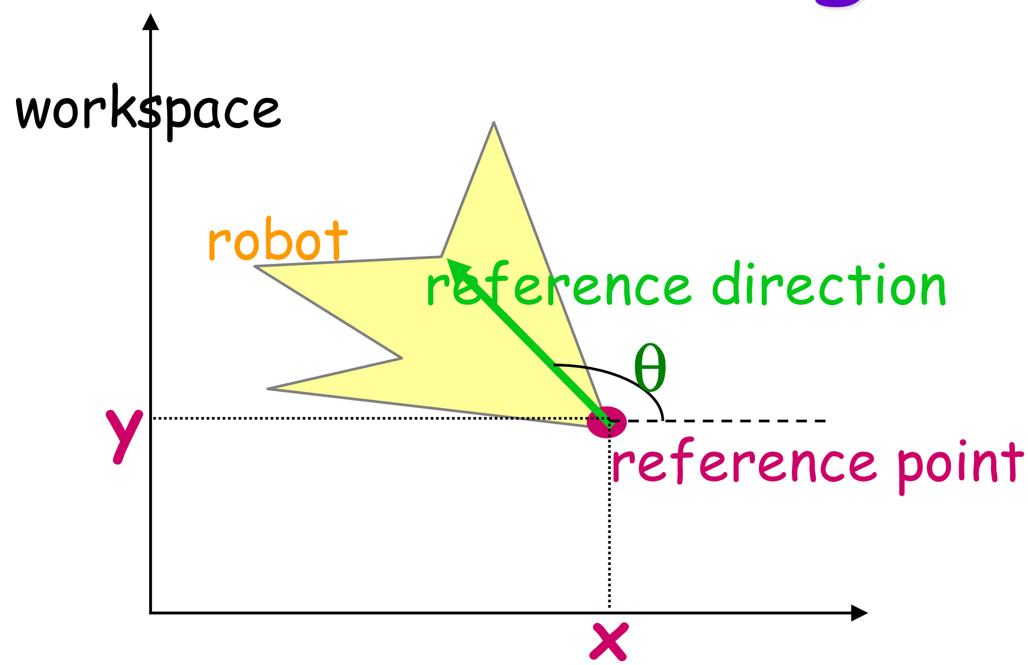
Structure of Configuration Space

- It is a manifold
  For each point q, there is a 1-to-1 map between a neighborhood of q and a Cartesian space $\mathbf{R}^n$, where n is the dimension of C
- This map is a local coordinate system called a chart.
  C can always be covered by a finite number of charts. Such a set is called an atlas

Example



Case of a Planar Rigid Robot



- 3-parameter representation: $q = (x,y,\theta)$ with $\theta \in [0,2\pi)$. Two charts are needed
- Other representation: $q = (x,y,\cos\theta,\sin\theta)$
  → c-space is a 3-D cylinder $\mathbf{R}^2 \times S^1$
  embedded in a 4-D space

Rigid Robot in 3-D Workspace

- $q = (x,y,z,\alpha,\beta,\gamma)$

The c-space is a 6-D space (manifold) embedded in a 12-D Cartesian space. It is denoted by $R^3xSO(3)$

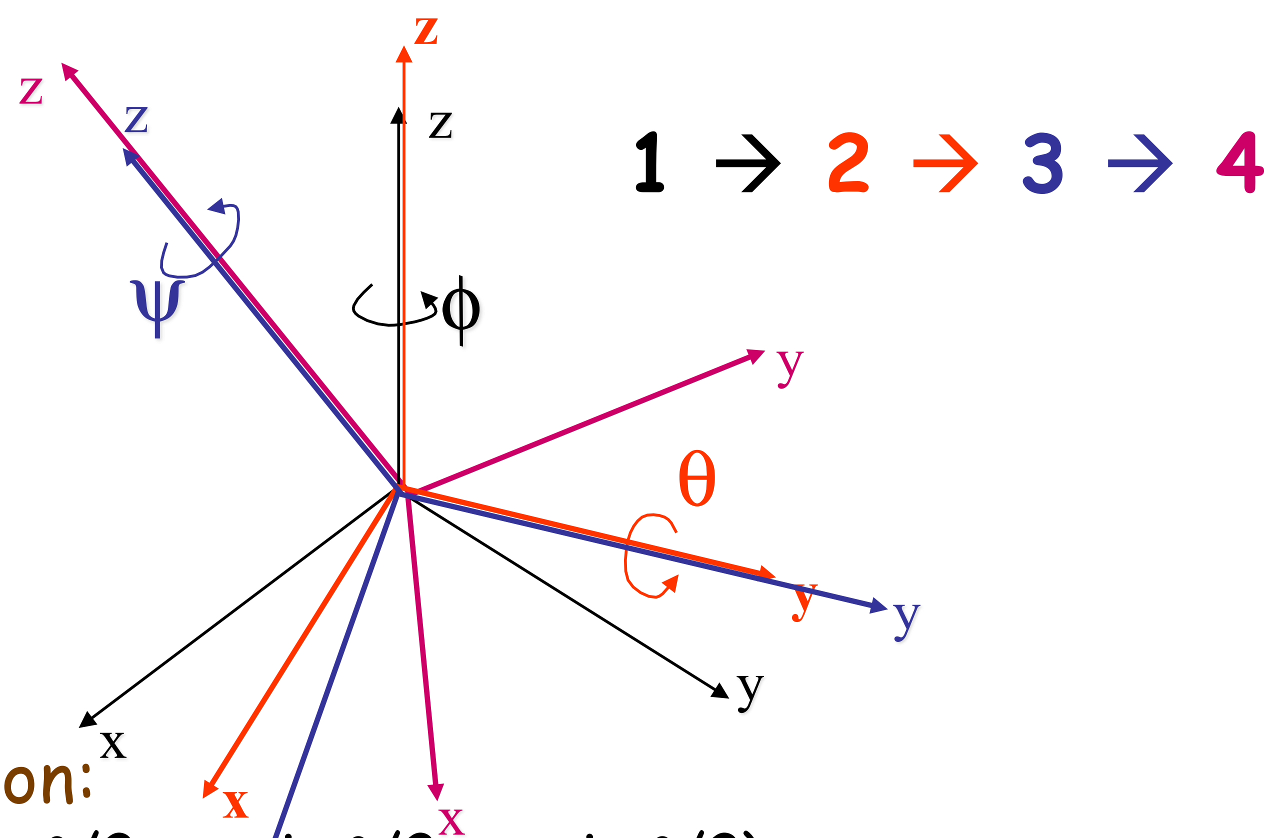
- Other representation: $q = (x,y,z,r_{11},r_{12},\ldots,r_{33})$ where r_{11}, r_{12}, …, r_{33} are the elements of rotation matrix R:

$$\begin{pmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{pmatrix}$$

with:
  - $r_{i1}^2+r_{i2}^2+r_{i3}^2 = 1$
  - $r_{i1}r_{j1} + r_{i2}r_{2j} + r_{i3}r_{j3} = 0$
  - $\det(R) = +1$

Parameterization of SO(3)

- Euler angles: (ϕ,θ,ψ)



- Unit quaternion:

  $(\cos\theta/2,\ n_1 \sin\theta/2,\ n_2 \sin\theta/2,\ n_3 \sin\theta/2)$

Metric in Configuration Space

A metric or distance function d in C is a map

$$d: (q_1,q_2) \in C^2 \rightarrow d(q_1,q_2) \geq 0$$

such that:

- $d(q_1,q_2) = 0$ if and only if $q_1 = q_2$
- $d(q_1,q_2) = d(q_2,q_1)$
- $d(q_1,q_2) \leq d(q_1,q_3) + d(q_3,q_2)$

Metric in Configuration Space

Example:

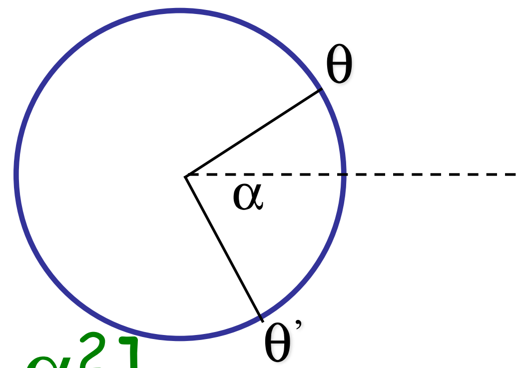
- Robot ***A*** and point x of ***A***
- x(q): location of x in the workspace when *A* is at configuration q
- *A* distance d in *C* is defined by:

$$d(q,q') = \max_{x \in A} ||x(q)-x(q')||$$

where ||a - b|| denotes the Euclidean distance between points a and b in the workspace

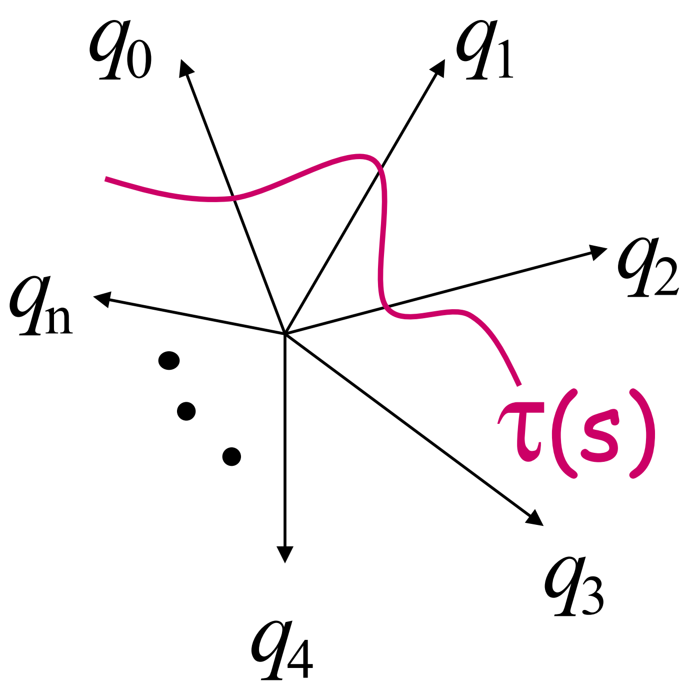
Specific Examples in $R^2 \times S^1$

- $q = (x,y,\theta)$, $q' = (x',y',\theta')$ with $\theta, \theta' \in [0,2\pi)$
- $\alpha = \min\{|\theta-\theta'|, 2\pi-|\theta-\theta'|\}$



- $d(q,q') = \mathrm{sqrt}[(x-x')^2 + (y-y')^2 + \alpha^2]$
- $d(q,q') = \mathrm{sqrt}[(x-x')^2 + (y-y')^2 + (\alpha\rho)^2]$ where ρ is the maximal distance between the reference point and a robot point

Notion of a Path

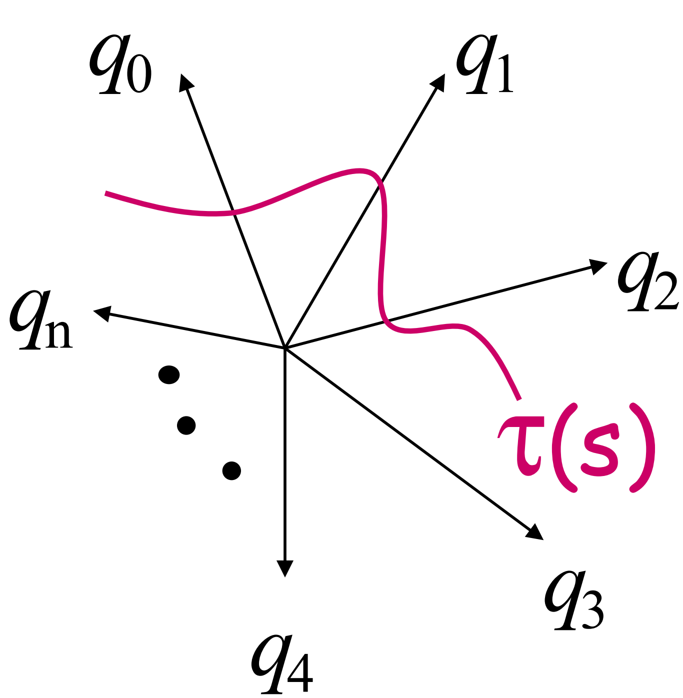


- A path in C is a piece of continuous curve connecting two configurations q and q':

$$\tau : s \in [0,1] \rightarrow \tau(s) \in C$$

- $s' \rightarrow s \quad \Rightarrow d(\tau(s),\tau(s')) \rightarrow 0$

Other Possible Constraints on Path



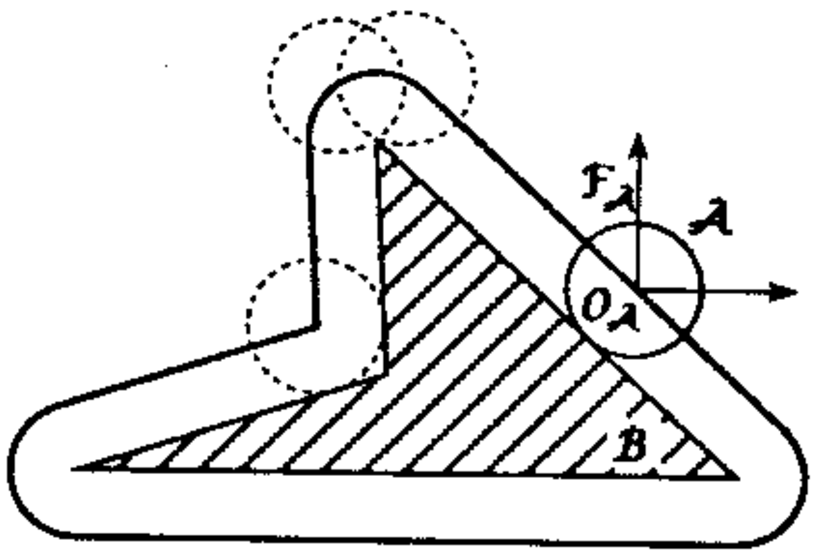
- Finite length, smoothness, curvature, etc…
- A trajectory is a path parameterized by time:

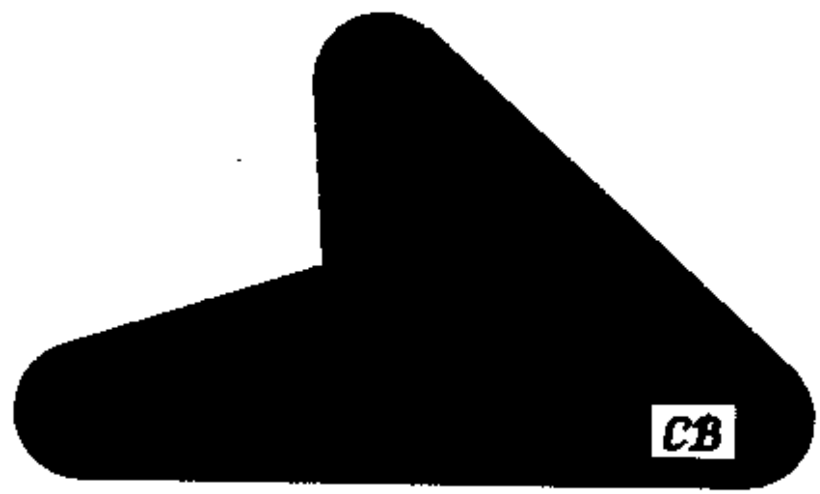
$$\tau : t \in [0,T] \rightarrow \tau(t) \in C$$

Obstacles in C-Space

- A configuration q is collision-free, or free, if the robot placed at q has null intersection with the obstacles in the workspace
- The free space F is the set of free configurations
- A C-obstacle is the set of configurations where the robot collides with a given workspace obstacle
- A configuration is semi-free if the robot at this configuration touches obstacles without overlap

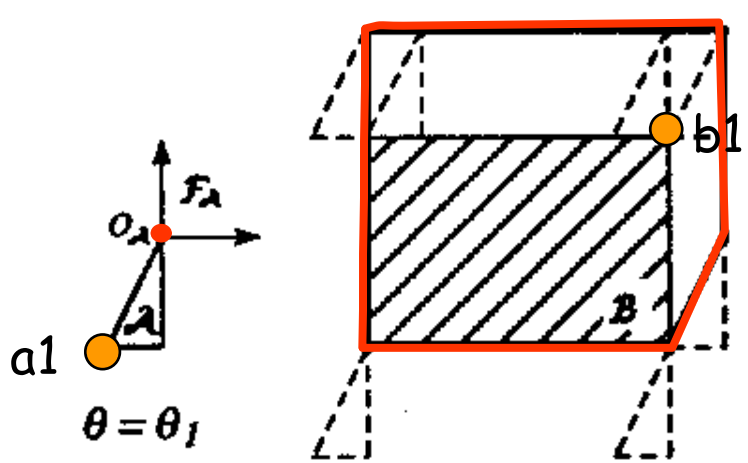
Disc Robot in 2-D Workspace

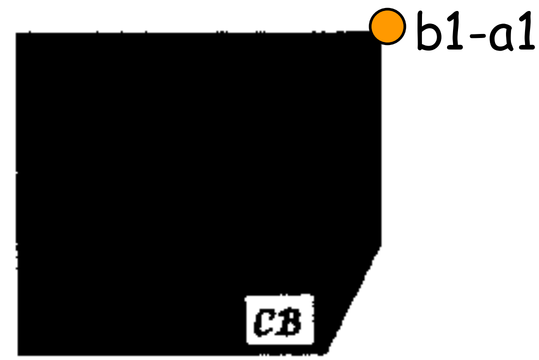


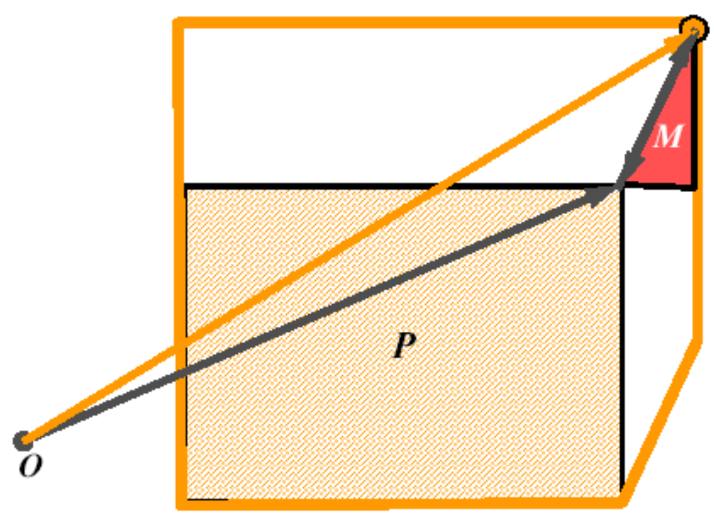


Rigid Robot Translating in 2-D

$$CB = B \ominus A = \{b - a \mid a \in A,\ b \in B\}$$







Rigid Robot Translating in 2-D

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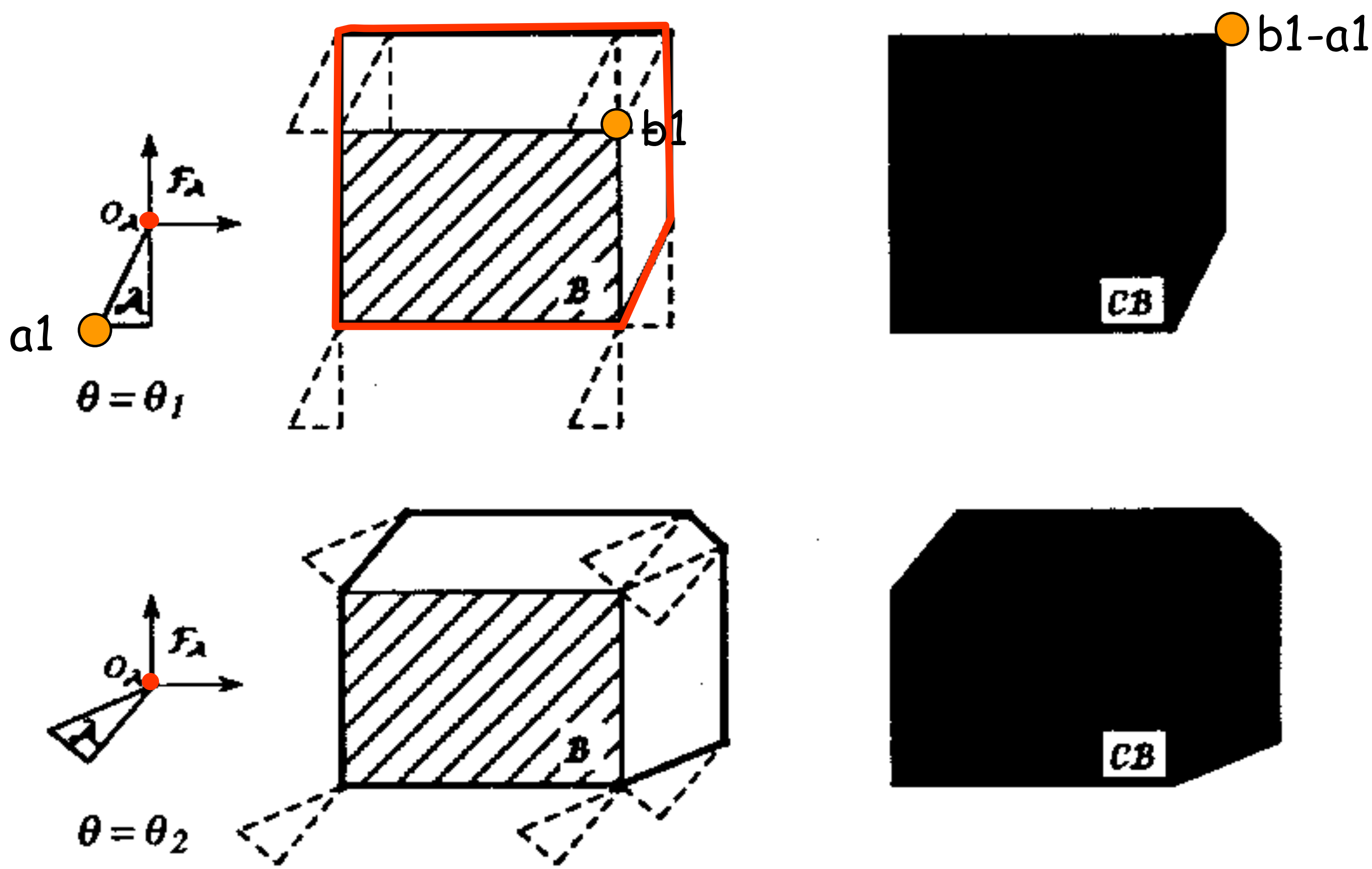
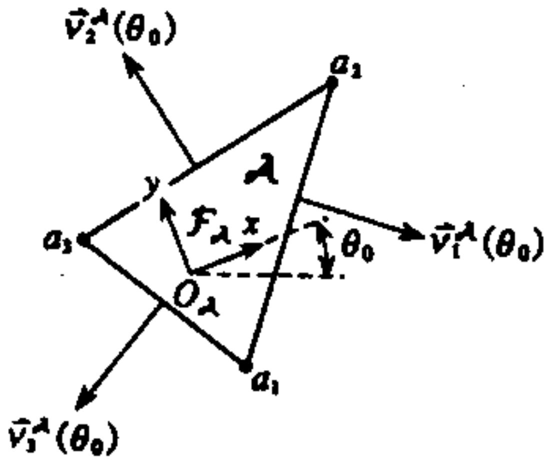
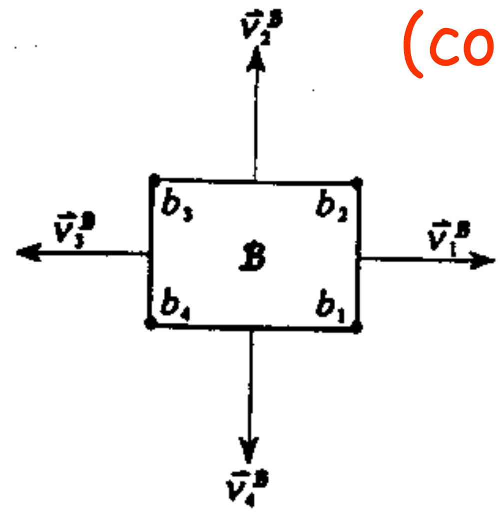


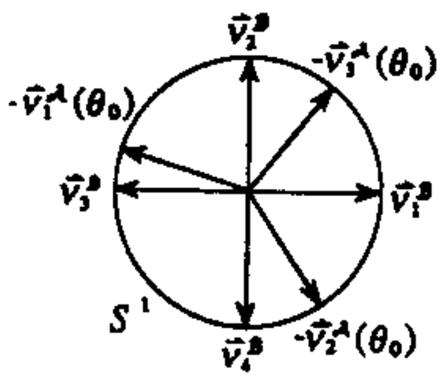
Linear-Time Computation of C-Obstacle in 2-D

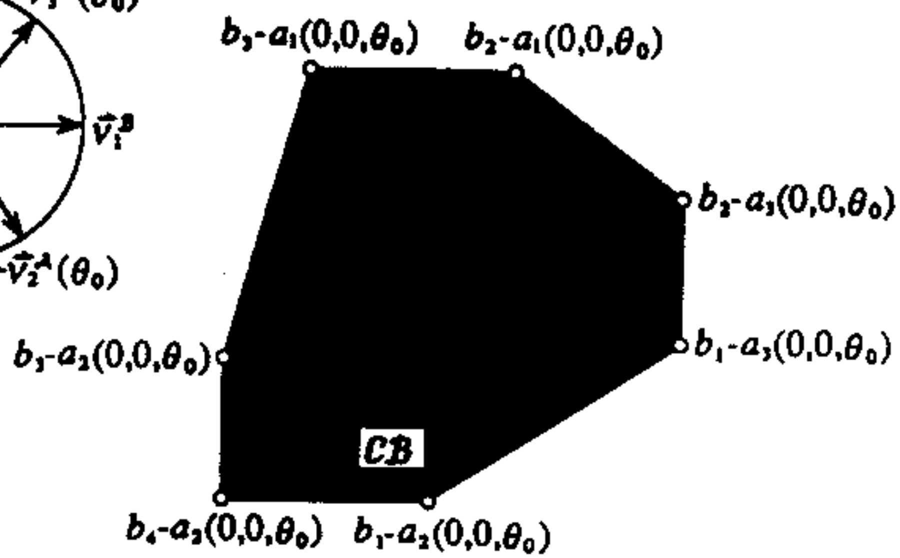
(convex polygons)



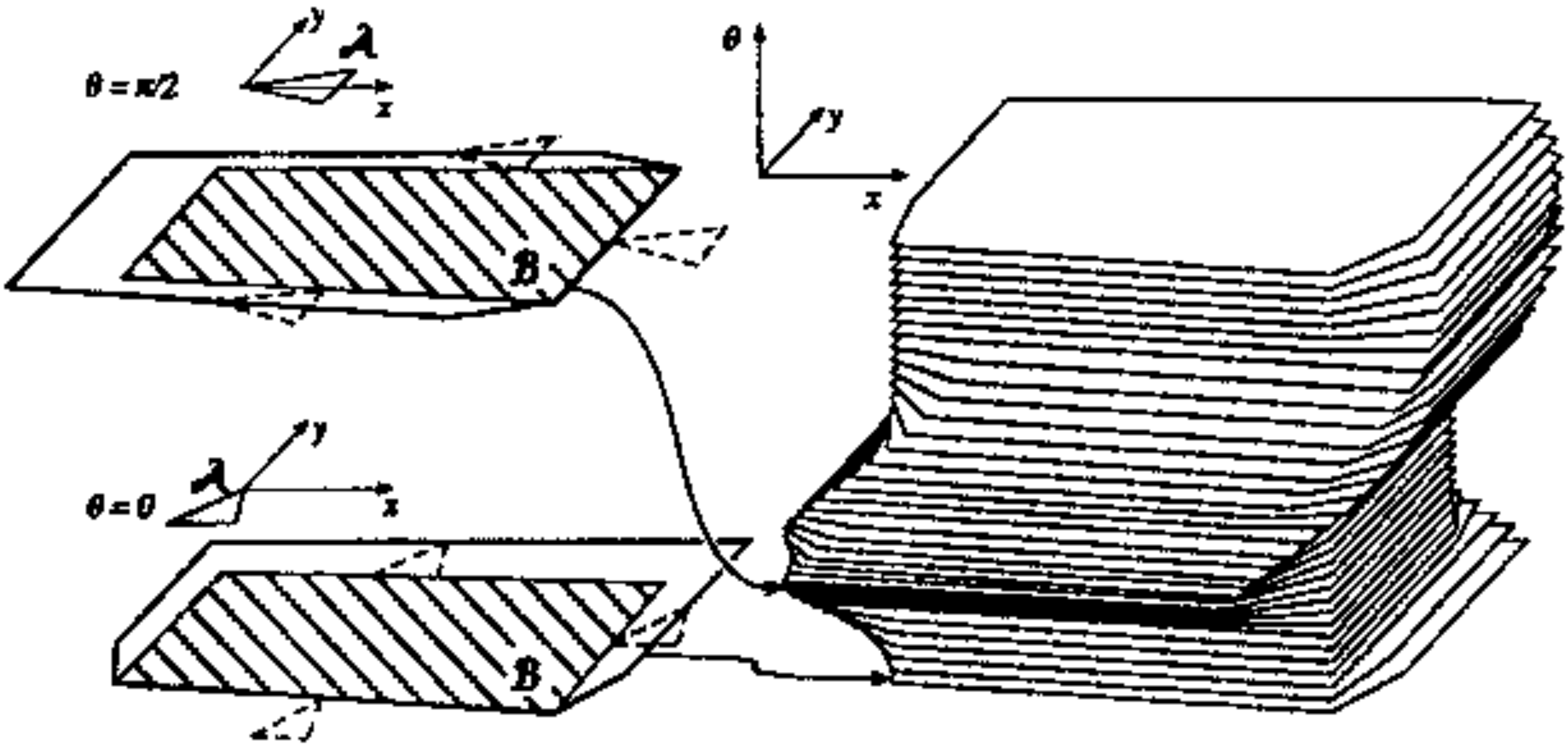


O(n+m)

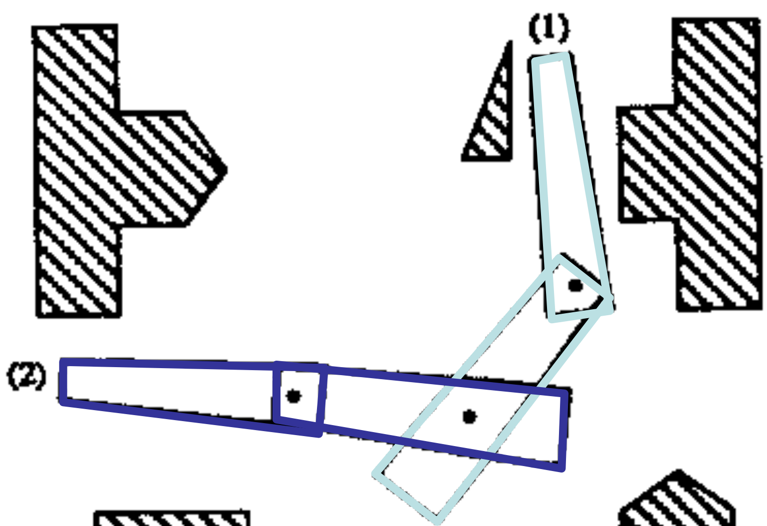


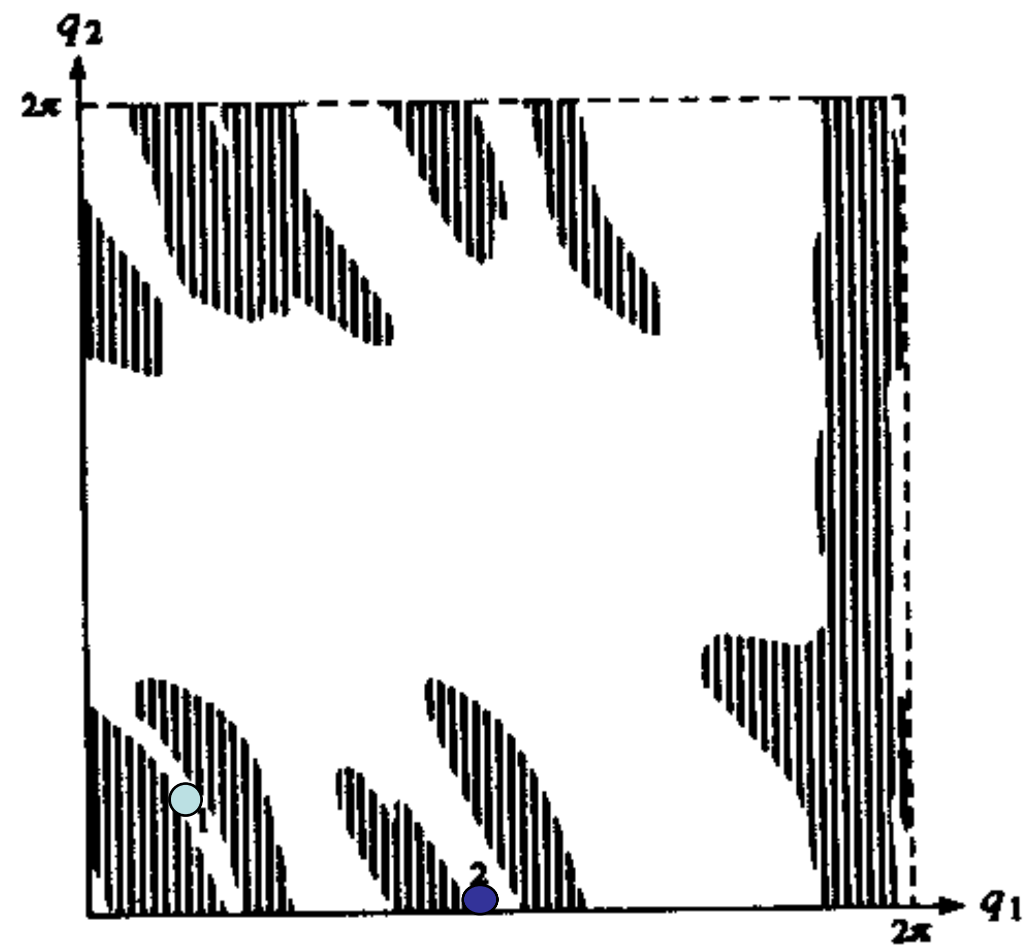


Rigid Robot Translating and Rotating in 2-D



C-Obstacle for Articulated Robot



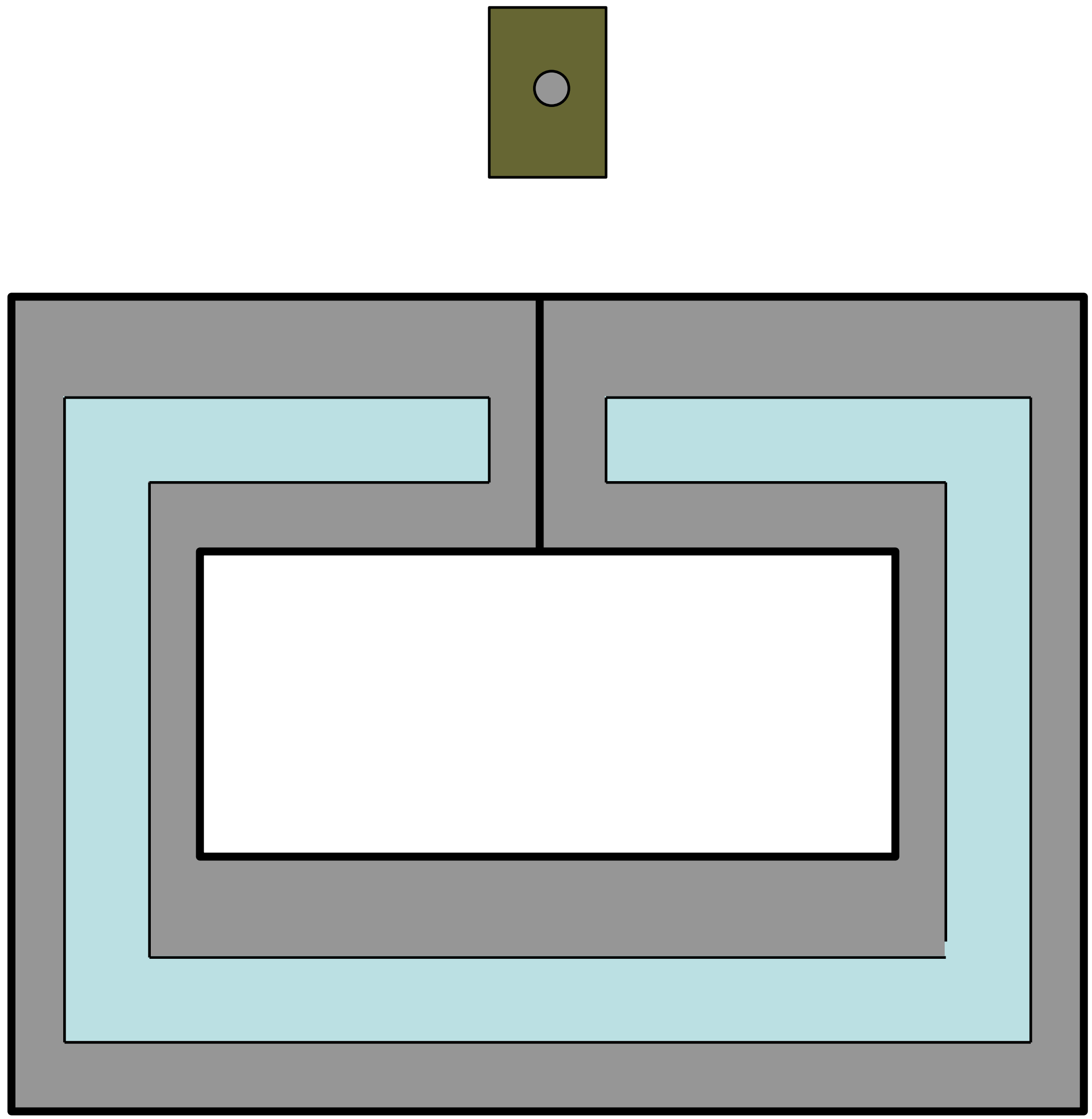


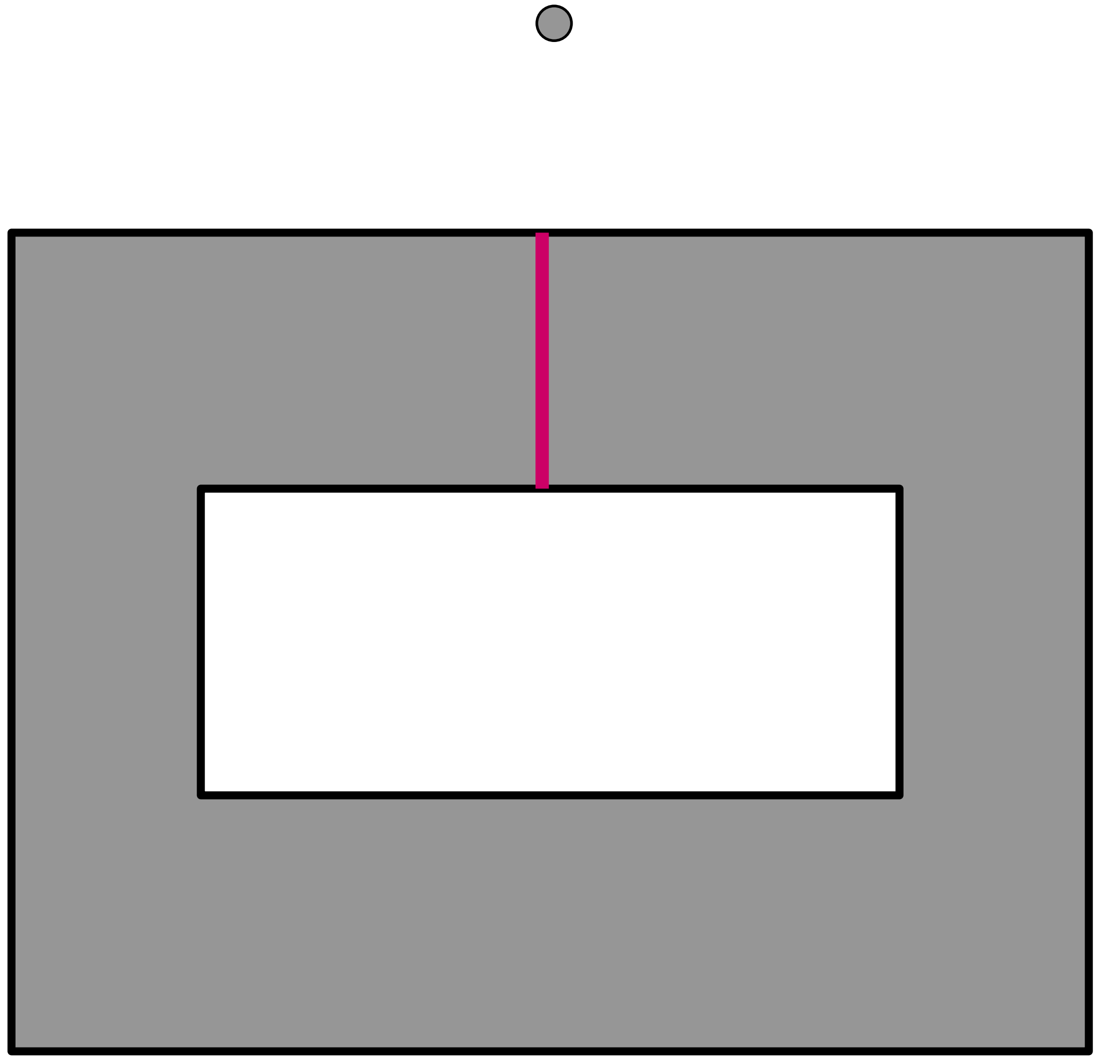
Free and Semi-Free Paths

- A free path lies entirely in the free space F
- A semi-free path lies entirely in the semi-free space

Remark on Free-Space Topology

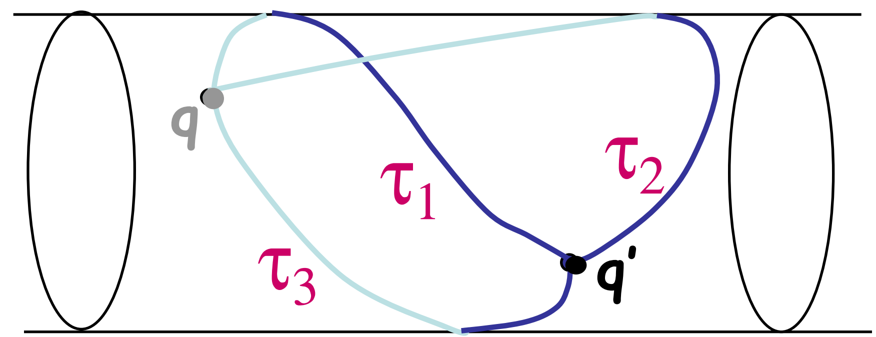
- The robot and the obstacles are modeled as closed subsets, meaning that they contain their boundaries
- One can show that the *C*-obstacles are closed subsets of the configuration space *C* as well
- Consequently, the free space F is an open subset of *C*. Hence, each free configuration is the center of a ball of non-zero radius entirely contained in F
- The semi-free space is a closed subset of *C*. Its boundary is a superset of the boundary of F





Notion of Homotopic Paths

- Two paths with the same endpoints are homotopic if one can be continuously deformed into the other
- R x S^1 example:



- τ_1 and τ_2 are homotopic
- τ_1 and τ_3 are not homotopic
- In this example, infinity of homotopy classes

Connectedness of C-Space

- *C* is connected if every two configurations can be connected by a path
- *C* is simply-connected if any two paths connecting the same endpoints are homotopic
  Examples: $\mathbf{R}^2$ or $\mathbf{R}^3$
- Otherwise *C* is multiply-connected
  Examples: S^1 and SO(3) are multiply- connected:
  - In S^1, infinity of homotopy classes
  - In SO(3), only two homotopy classes